

A characterization of nilpotent orbit closures among symplectic singularities

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Symplectic singularities have been playing important roles both in algebraic geometry and geometric representation theory ever since Beauville introduced their notion in [Be]. Most examples of symplectic singularities admit natural \mathbf{C}^* -actions with only positive weights. Kaledin [Ka] conjectured that any symplectic singularity admits such a \mathbf{C}^* -action.

If a symplectic singularity has a \mathbf{C}^* -action with positive weights, it can be globalized to an affine variety with a \mathbf{C}^* -action. Such an affine variety is called a *conical symplectic variety*. More precisely, an affine normal variety $X = \text{Spec } R$ is a conical symplectic variety if

- (i) R is positively graded: $R = \bigoplus_{i \geq 0} R_i$ with $R_0 = \mathbf{C}$;
- (ii) the smooth part X_{reg} admits a homogeneous symplectic 2-form ω and it extends to a holomorphic 2-form on a resolution \tilde{X} of X .

Denote the \mathbf{C}^* -action by $t : X \rightarrow X$ ($t \in \mathbf{C}^*$). By the assumption we have $t^*\omega = t^l\omega$ for some integer l . This integer l is called the weight of ω and is denoted by $wt(\omega)$. By the extension property (ii) we have $wt(\omega) > 0$ (cf. [Na 3], Lemma (2.2)).

Let $\{x_0, \dots, x_n\}$ be a set of minimal homogeneous generators of the \mathbf{C} -algebra R and put $a_i := \deg x_i$. We put $N := \max\{a_0, \dots, a_n\}$ and call N the *maximal weight* of X . It is uniquely determined by a conical symplectic variety X . By [Na 1], there are only finitely many conical symplectic varieties (X, ω) of a fixed dimension $2d$ and with a fixed maximal weight N , up to an isomorphism. In this sense it would be important to classify conical symplectic varieties with maximal weight 1. By the homogeneous generators $\{x_i\}$, we can embed X into an affine space \mathbf{C}^{n+1} . In [Na 2] we treat the case where $X \subset \mathbf{C}^{n+1}$ is a complete intersection of homogeneous polynomials. The main theorem of [Na 2] asserts that (X, ω) is isomorphic to the nilpotent cone (N, ω_{KK}) of a complex semisimple Lie algebra \mathfrak{g} together with the Kirillov-Kostant 2-form provided that X is singular. However, there are a lot of examples of maximal weight 1 which are not of complete intersection. In fact, a nilpotent orbit O of a complex semisimple Lie algebra \mathfrak{g} admits the Kirillov-Kostant form ω_{KK} and if its closure \bar{O} is normal, then (\bar{O}, ω_{KK}) is a conical symplectic variety with maximal weight 1 by Panyushev [Pa] and Hinich [Hi].

A main purpose of this article is to prove that they actually exhaust all conical symplectic varieties with maximal weight 1.

Theorem. *Let (X, ω) be a conical symplectic variety with maximal weight 1. Then (X, ω) is isomorphic to one of the following:*

- (i) $(\mathbf{C}^{2d}, \omega_{st})$ with $\omega_{st} = \sum_{1 \leq i \leq d} dz_i \wedge dz_{i+d}$,
- (ii) (\bar{O}, ω_{KK}) where \bar{O} is a normal nilpotent orbit closure of a complex semisimple Lie algebra \mathfrak{g} and ω_{KK} is the Kirillov-Kostant form.

There is a non-normal nilpotent orbit closure in a complex semisimple Lie algebra. The normalization \tilde{O} of such an orbit closure \bar{O} is also a conical symplectic variety.¹ But the maximal weight of \tilde{O} is usually larger than 1.²

We first notice that ω determines a Poisson structure on X_{reg} in a usual way. By the normality of X , it uniquely extends to a Poisson structure $\{\cdot, \cdot\} : O_X \times O_X \rightarrow O_X$. In particular, R becomes a Poisson \mathbf{C} -algebra with a Poisson bracket of degree $-wt(\omega)$.

In the remainder, X is a conical symplectic variety with the maximal weight $N = 1$. First of all, we prove in Proposition 1 that $wt(\omega) = 2$ or $wt(\omega) = 1$. In the first case (X, ω) is isomorphic to an affine space \mathbf{C}^{2d} together with the standard symplectic form ω_{st} . In the second case the Poisson bracket has degree -1 and R_1 has a natural Lie algebra structure. Then it is fairly easy to show that X is a coadjoint orbit closure of a complex Lie algebra \mathfrak{g} (Proposition 3). If X has a crepant resolution, we can prove that \mathfrak{g} is semisimple in the same way as in [Na 2]. But X generally does not have such a resolution and we need a new method to prove the semisimplicity. This is nothing but Proposition 4.

Proposition 1 *Assume that X is a conical symplectic variety with maximal weight $N = 1$. Then $wt(\omega) = 1$ or $wt(\omega) = 2$. If $wt(\omega) = 2$, then (X, ω) is isomorphic to an affine space $(\mathbf{C}^{2d}, \omega_{st})$ with the standard symplectic form.*

Remark. As is remarked in the beginning of [Na 2, §2], if X is a smooth conical symplectic variety with maximal weight 1, then $(X, \omega) \cong (\mathbf{C}^{2d}, \omega_{st})$. Hence X is singular exactly when $wt(\omega) = 1$.

Proof. Since $N = 1$, the coordinate ring R is generated by R_1 . We put $l := wt(\omega)$. We already know that $l > 0$. If $l > 2$, then we have $\{R_1, R_1\} = 0$ and hence $\{R, R\} = 0$, which is absurd. We now assume that $l = 2$ and prove that X is an affine space with the standard symplectic form. Then the Poisson bracket induces a skew-symmetric form $R_1 \times R_1 \rightarrow R_0 = \mathbf{C}$. If this is a degenerate skew-symmetric form, then we can choose a non-zero element $x_1 \in R_1$ such that $\{x_1, \cdot\} = 0$. Notice that $x_1 = 0$ determines a non-zero effective divisor D on X_{reg} . If we choose a general point $a \in D$, then the reduced divisor D_{red} is smooth around a . Consider an analytic open neighborhood $U \subset X_{reg}$ of a . Then there is a system of local coordinates $\{z_1, \dots, z_{2d}\}$ of U such that x_1 can be written as $x_1 = z_1^m$ for a suitable $m > 0$. The Poisson structure on X induces a non-degenerate Poisson structure $\{\cdot, \cdot\}_U$ on U . But, by the choice of x_1 , we have $\{z_1^m, \cdot\}_U = m z_1^{m-1} \{z_1, \cdot\}_U = 0$, which implies that $\{z_1, \cdot\}_U = 0$. This contradicts that the Poisson bracket $\{\cdot, \cdot\}_U$ is non-degenerate.

Therefore the skew-symmetric form is non-degenerate. In this case X is a closed Poisson subscheme of an affine space with a non-degenerate Poisson structure induced by the standard symplectic form. But such an affine space has no Poisson closed subscheme

¹By [K-P, Proposition 7.4] \bar{O} is always resolved by a vector bundle Y over G/P with a parabolic subgroup P of the adjoint group G of \mathfrak{g} . Denote this resolution by $\pi : Y \rightarrow \bar{O}$. The map π factorizes as $Y \rightarrow \tilde{O} \rightarrow \bar{O}$. The fiber $\pi^{-1}(0)$ coincides with the zero section of Y , which is isomorphic to G/P . As G/P is connected, the fibre $\mu^{-1}(0)$ of the normalization map $\mu : \tilde{O} \rightarrow \bar{O}$ consists of just one point, say $x \in \tilde{O}$. The \mathbf{C}^* -action on \tilde{O} extends to a \mathbf{C}^* -action on \bar{O} with a unique fixed point x . It is easily checked that this \mathbf{C}^* -action has only positive weights and \tilde{O} becomes a conical symplectic variety.

²It may happen that \tilde{O} coincides with a normal nilpotent orbit closure of a different complex semisimple Lie algebra. In such a case the maximal weight is 1.

except the affine space itself. Therefore $X = \text{Spec} R$. Q.E.D.

The regular part X_{reg} of a conical symplectic variety X is a smooth Poisson variety. Let $\Theta_{X_{reg}}$ denote the sheaf of vector fields on X_{reg} . By using the Poisson bracket we define the Lichnerowicz-Poisson complex

$$0 \rightarrow \Theta_{X_{reg}} \xrightarrow{\delta_1} \wedge^2 \Theta_{X_{reg}} \xrightarrow{\delta_2} \dots$$

by

$$\begin{aligned} \delta_p f(da_1 \wedge \dots \wedge da_{p+1}) &:= \sum_{i=1}^{p+1} (-1)^{i+1} \{a_i, f(da_1 \wedge \dots \wedge \hat{da}_i \wedge \dots \wedge da_{p+1})\} \\ &+ \sum_{j < k} (-1)^{j+k} f(d\{a_j, a_k\} \wedge da_1 \wedge \dots \wedge \hat{da}_j \wedge \dots \wedge \hat{da}_k \wedge \dots \wedge da_{p+1}). \end{aligned}$$

In the Lichnerowicz-Poisson complex, $\wedge^p \Theta_{X_{reg}}$ is placed in degree p . The Lichnerowicz-Poisson complex of X_{reg} is closely related to the Poisson deformation of $(X, \{, \})$. For details, see [Na 4].

In the remainder we assume that $wt(\omega) = 1$. The Poisson bracket then defines a pairing map $R_1 \times R_1 \rightarrow R_1$ and R_1 becomes a Lie algebra. We denote by \mathfrak{g} this Lie algebra. As all generators have weight 1, we have a surjection $\oplus \text{Sym}^i(R_1) \rightarrow R$. It induces a \mathbf{C}^* -equivariant closed embedding $X \rightarrow \mathfrak{g}^*$.

Recall that the adjoint group G of \mathfrak{g} (cf. [Pro, p.86]) is defined as a subgroup of $GL(\mathfrak{g})$ generated by all elements of the form $\exp(ad v)$ with $v \in \mathfrak{g}$. The adjoint group G is a complex Lie subgroup of $GL(\mathfrak{g})$, but it is not necessarily a closed algebraic subgroup of $GL(\mathfrak{g})$. Moreover, the Lie algebra $Lie(G)$ does not necessarily coincide with \mathfrak{g} . We have $Lie(G) = \mathfrak{g}$ if and only if the adjoint representation is a faithful \mathfrak{g} -representation, or equivalently, \mathfrak{g} has trivial center.

Proposition 2. *Let $\text{Aut}^{\mathbf{C}^*}(X, \omega)$ denote the \mathbf{C}^* -equivariant automorphism group preserving ω . Then the identity component of $\text{Aut}^{\mathbf{C}^*}(X, \omega)$ can be identified with the adjoint group G of \mathfrak{g} . Moreover \mathfrak{g} has trivial center. In particular, \mathfrak{g} is the Lie algebra of the linear algebraic group $\text{Aut}^{\mathbf{C}^*}(X, \omega)$.*

Proof. Let $(\wedge^{\geq 1} \Theta_{X_{reg}}, \delta)$ be the Lichnerowicz-Poisson complex for the smooth Poisson variety X_{reg} . The algebraic torus \mathbf{C}^* acts on $\Gamma(X_{reg}, \wedge^p \Theta_{X_{reg}})$ and there is an associated grading

$$\Gamma(X_{reg}, \wedge^p \Theta_{X_{reg}}) = \oplus_{n \in \mathbf{Z}} \Gamma(X_{reg}, \wedge^p \Theta_{X_{reg}})(n).$$

Since the Poisson bracket of X has degree -1 , the coboundary map δ has degree -1 ; thus we have a complex

$$\Gamma(X_{reg}, \Theta_{X_{reg}})(0) \xrightarrow{\delta_1} \Gamma(X_{reg}, \wedge^2 \Theta_{X_{reg}})(-1) \xrightarrow{\delta_2} \dots$$

The kernel $\text{Ker}(\delta_1)$ of this complex is isomorphic to the tangent space of $\text{Aut}^{\mathbf{C}^*}(X, \omega)$ at $[id]$. In fact, an element of $\text{Ker}(\delta_1)$ corresponds to a derivation of $\mathcal{O}_{X_{reg}}$ (or an infinitesimal automorphism of X_{reg}) preserving the Poisson structure, but it uniquely

extends to a derivation of O_X preserving the Poisson structure (cf. [Na 4, Proposition 8]).

The Lichnerowicz-Poisson complex $(\wedge^{\geq 1}\Theta_{X_{reg}}, \delta)$ is identified with the truncated De Rham complex $(\Omega_{X_{reg}}^{\geq 1}, d)$ by the symplectic form ω (cf. [Na 4, Proposition 9], [Na 3, Section 3]). The algebraic torus \mathbf{C}^* acts on $\Gamma(X_{reg}, \Omega_{X_{reg}}^p)$ and there is an associated grading

$$\Gamma(X_{reg}, \Omega_{X_{reg}}^p) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X_{reg}, \Omega_{X_{reg}}^p)(n).$$

The coboundary map d has degree 0; thus we have a complex

$$\Gamma(X_{reg}, \Omega_{X_{reg}}^1)(1) \xrightarrow{d_1} \Gamma(X_{reg}, \Omega_{X_{reg}}^2)(1) \xrightarrow{d_2} \dots$$

Since ω has weight 1, this complex is identified with the the Lichnerowicz-Poisson complex above.

There is an injective map $d : \Gamma(X_{reg}, O_{X_{reg}})(1) \rightarrow \Gamma(X_{reg}, \Omega_{X_{reg}}^1)(1)$.

We shall prove that $\text{Ker}(d_1) = \Gamma(X_{reg}, O_{X_{reg}})(1)$. The \mathbf{C}^* -action on X defines a vector field ζ on X_{reg} . For $v \in \Gamma(X_{reg}, \Omega_{X_{reg}}^1)(1)$, the Lie derivative $L_\zeta v$ of v along ζ equals v . If moreover v is d -closed, then one has $v = d(i_\zeta v)$ by the Cartan relation

$$L_\zeta v = d(i_\zeta v) + i_\zeta(dv).$$

This means that $v \in \Gamma(X_{reg}, O_{X_{reg}})(1)$. On the other hand, we have $\Gamma(X_{reg}, O_{X_{reg}})(1) = \Gamma(X, O_X)(1) = R_1 = \mathfrak{g}$.

It follows from the identification of $\text{Ker}(\delta_1)$ and $\text{Ker}(d_1)$ that every element of $\text{Ker}(\delta_1)$ is a Hamiltonian vector field $H_f := \{f, \cdot\}$ for some $f \in R_1$. In particular, for $g \in R_1$, we have $H_f(g) = [f, g]$. Since $H_f \neq 0$ for a non-zero f , the map $ad : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ is an injection. Notice that an element of $\text{Aut}^{\mathbf{C}^*}(X, \omega)$ determines an automorphism of a graded \mathbf{C} -algebra R . In particular, it induces a \mathbf{C} -linear automorphism of $R_1 = \mathfrak{g}$. Since R is generated by R_1 , this linear automorphism completely determines an automorphism of R . Hence, both G and $\text{Aut}^{\mathbf{C}^*}(X, \omega)$ are subgroups of $GL(\mathfrak{g})$. The tangent spaces of both subgroups at $[id]$ coincide with $\mathfrak{g} \cong ad(\mathfrak{g}) \subset \text{End}(\mathfrak{g})$. Therefore G is the identity component of $\text{Aut}^{\mathbf{C}^*}(X, \omega)$ and $\text{Lie}(G) = \mathfrak{g}$. Q.E.D.

Proposition 3 *The symplectic variety X coincides with the closure of a coadjoint orbit of \mathfrak{g}^* .*

Proof. Since G is the identity component of $\text{Aut}^{\mathbf{C}^*}(X, \omega)$, X is stable under the coadjoint action of G on \mathfrak{g} . Hence X is a union of G -orbits. The G -orbits in X are symplectic leaves of the Poisson variety X . In our case, since X has only symplectic singularities, X has only finitely many symplectic leaves by [Ka]. Therefore X consists of finite number of G -orbits; hence there is an open dense G -orbit and X is the closure of such an orbit. Q.E.D.

For the unipotent radical U of G , let us denote by \mathfrak{n} its Lie algebra ³. Assume that $\mathfrak{n} \neq 0$. Then the center $z(\mathfrak{n})$ of \mathfrak{n} is also non-trivial because \mathfrak{n} is a nilpotent Lie algebra.

³The ideal \mathfrak{n} is actually the nilradical of \mathfrak{g} when \mathfrak{g} has trivial center.

Moreover $z(\mathfrak{n})$ is an ideal of \mathfrak{g} . In fact, it is enough to prove that, if $y \in \mathfrak{g}$ and $z \in z(\mathfrak{n})$, then $[x, [y, z]] = 0$ for any $x \in \mathfrak{n}$. Consider the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

First, since $z \in z(\mathfrak{n})$, one has $[z, x] = 0$. Next, since \mathfrak{n} is an ideal of \mathfrak{g} , we have $[x, y] \in \mathfrak{n}$; hence $[z, [x, y]] = 0$. It then follows from the Jacobi identity that $[x, [y, z]] = 0$.

Proposition 4. *Let \mathfrak{g} be a complex Lie algebra with trivial center whose adjoint group G is a linear algebraic group. Assume that $\mathfrak{n} \neq 0$. Let O be a coadjoint orbit of \mathfrak{g}^* with the following properties*

- (i) *O is preserved by the scalar \mathbf{C}^* -action on \mathfrak{g}^* ;*
- (ii) *$T_0\bar{O} = \mathfrak{g}^*$, where $T_0\bar{O}$ denotes the tangent space of the closure \bar{O} of O at the origin.*

Then $\bar{O} - O$ contains infinitely many coadjoint orbits; in particular \bar{O} has infinitely many symplectic leaves.

Remark. This proposition shows that \bar{O} cannot have symplectic singularities. In fact, if \bar{O} has symplectic singularities, it has only finitely many symplectic leaves by [Ka].

Proof of Proposition 4. By a result of Mostow [Mos] (cf. [Ho], VIII, Theorem 3.5, Theorem 4.3), G is a semi-direct product of a reductive subgroup L and the unipotent radical U . Therefore we have a decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}$. Take an element $\phi \in O$. Then ϕ is a linear function on \mathfrak{g} , which restricts to a *non-zero* function on $z(\mathfrak{n})$. In fact, if ϕ is zero on $z(\mathfrak{n})$, then $O \subset (\mathfrak{g}/z(\mathfrak{n}))^*$ and hence $\bar{O} \subset (\mathfrak{g}/z(\mathfrak{n}))^*$, which contradicts the assumption (ii). We put $\bar{\phi} := \phi|_{z(\mathfrak{n})} \neq 0$.

Notice that the adjoint group G is the subgroup of $\mathrm{GL}(\mathfrak{g})$ generated by all elements of the form $\exp(\mathrm{ad} v)$ with $v \in \mathfrak{g}$. If $v \in z(\mathfrak{n})$, then $\exp(\mathrm{ad} v) = \mathrm{id} + \mathrm{ad} v$ because $(\mathrm{ad} v)^2 = 0$ for $v \in z(\mathfrak{n})$. Let $Z(U)$ be the identity component of the center of the unipotent radical U . Then one can write $Z(U) = 1 + \mathrm{ad} z(\mathfrak{n})$. By the assumption, the map $\mathrm{ad} : \mathfrak{g} \rightarrow \mathrm{End}(\mathfrak{g})$ is an injection. Now we identify $z(\mathfrak{n})$ with $\mathrm{ad} z(\mathfrak{n})$; then one can write $Z(U) = 1 + z(\mathfrak{n})$ and its group law is defined by $(1 + v)(1 + v') := 1 + (v + v')$ for $v, v' \in z(\mathfrak{n})$. Fix an element $v \in z(\mathfrak{n})$ and consider $\mathrm{Ad}_{1+v}^*(\phi) \in \mathfrak{g}^*$. Since $(1+v)^{-1} = 1 - v$, the adjoint action

$$\mathrm{Ad}_{(1+v)^{-1}} : \mathfrak{l} \oplus \mathfrak{n} \rightarrow \mathfrak{l} \oplus \mathfrak{n}$$

is defined by

$$x \oplus y \rightarrow x \oplus (-[v, x] + y)$$

because $(\mathrm{ad} v)(y) = 0$. By definition

$$\mathrm{Ad}_{1+v}^*(\phi)(x \oplus y) = \phi(\mathrm{Ad}_{(1+v)^{-1}}(x \oplus y)) = \phi(x \oplus y) - \bar{\phi}([v, x]).$$

Here notice that $[v, x] \in z(\mathfrak{n})$ and hence $\phi([v, x]) = \bar{\phi}([v, x])$.

Since $z(\mathfrak{n})$ is an \mathfrak{l} -module by the Lie bracket, we decompose it into irreducible factors $z(\mathfrak{n}) = \bigoplus V_i$. Notice that it is the same as the irreducible decomposition of $z(\mathfrak{n})$ as a $[\mathfrak{l}, \mathfrak{l}]$ -module if $[\mathfrak{l}, \mathfrak{l}] \neq 0$. In fact, the reductive Lie algebra \mathfrak{l} is written as a direct sum of the semi-simple part and the center: $\mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] \oplus z(\mathfrak{l})$. Since $z(\mathfrak{l})$ is an Abelian Lie algebra, $z(\mathfrak{n})$ can be written as a direct sum $\bigoplus V_\alpha$ of the weight spaces for $z(\mathfrak{l})$. The semisimple part $[\mathfrak{l}, \mathfrak{l}]$ acts on each weight space V_α ; hence V_α is a direct sum of irreducible

$[\mathfrak{l}, \mathfrak{l}]$ modules. These irreducible $[\mathfrak{l}, \mathfrak{l}]$ modules are stable under the $z(\mathfrak{l})$ -action and, hence are irreducible \mathfrak{l} -modules.

When $\dim V_i = 1$ for some i , this V_i is an ideal of \mathfrak{g} . By the assumption (i), one can write $\bar{O} = \text{Spec} R$ with a graded \mathbf{C} -algebra $R = \bigoplus_{j \geq 0} R_j$. By (ii) we see that $R_1 = \mathfrak{g}$. Take a generator x of a 1-dimensional space V_i . Then x generates a Poisson ideal I of R and $Y := \text{Spec}(R/I)$ is a closed Poisson subscheme of \bar{O} of codimension 1. Moreover, Y is stable under the G -action. Since $\dim Y$ is odd, Y contains infinitely many coadjoint orbits.

In the remainder we assume that $\dim V_i > 1$ for all i . In this case $[\mathfrak{l}, \mathfrak{l}] \neq 0$. Since $\bar{\phi} \neq 0$, we can choose an i such that $\phi|_{V_i} \neq 0$. We fix a Cartan subalgebra \mathfrak{h} of the semisimple Lie algebra $[\mathfrak{l}, \mathfrak{l}]$ and choose a set Δ of simple roots from the root system Φ . We define $\mathfrak{n}^+ := \bigoplus_{\alpha \in \Phi^+} [\mathfrak{l}, \mathfrak{l}]_\alpha$. Let $v_0 \in V_i$ be a highest weight vector of the irreducible $[\mathfrak{l}, \mathfrak{l}]$ -module V_i . Then one has $[v_0, \mathfrak{n}^+] = 0$ and, in particular, $\phi([v_0, \mathfrak{n}^+]) = 0$. Moreover, we may assume that $\bar{\phi}(v_0) \neq 0$ by replacing ϕ by a suitable $Ad_g^*(\phi)$ with $g \in L$. This is possible. In fact, if $Ad_g^*(\bar{\phi})(v_0) = 0$ for all g , then $\bar{\phi}$ is zero on the vector subspace of V_i spanned by all $Ad_g(v_0)$. But, since V_i is an irreducible L -representation, such a subspace coincides with V_i . This contradicts the fact that $\bar{\phi}|_{V_i} \neq 0$. Since v_0 is a highest weight vector of a non-trivial $[\mathfrak{l}, \mathfrak{l}]$ -irreducible module V_i , $[v_0, h]$ is a multiple of v_0 by a non-zero constant for an $h \in \mathfrak{h}$. Since $\bar{\phi}(v_0) \neq 0$, we also have $\bar{\phi}([v_0, h]) \neq 0$ for this $h \in \mathfrak{h}$.

Let us consider $\bar{\phi}_{v_0} := \bar{\phi}([v_0, \cdot])|_{[\mathfrak{l}, \mathfrak{l}]}$. By definition $\bar{\phi}_{v_0}$ is an element of $[\mathfrak{l}, \mathfrak{l}]^*$. By the Killing form it is identified with an element of $[\mathfrak{l}, \mathfrak{l}]$. The two facts $\bar{\phi}_{v_0}(\mathfrak{n}^+) = 0$ and $\bar{\phi}_{v_0}(h) \neq 0$ mean that $\bar{\phi}_{v_0}$ is *not* a nilpotent element of $[\mathfrak{l}, \mathfrak{l}]$.

For such v_0 and ϕ , we consider $Ad_{1+t^{-1}v_0}^*(t\phi)$, with $t \in \mathbf{C}^*$. One can write

$$Ad_{1+t^{-1}v_0}^*(t\phi)(x \oplus y) = t\phi(x \oplus y) - \bar{\phi}([v_0, x]).$$

Thus one has

$$\lim_{t \rightarrow 0} Ad_{1+t^{-1}v_0}^*(t\phi)(x \oplus y) = -\bar{\phi}([v_0, x]).$$

By definition $Ad_{1+t^{-1}v_0}^*(t\phi) \in O$. Thus $\lim_{t \rightarrow 0} Ad_{1+t^{-1}v_0}^*(t\phi) \in \bar{O}$. Moreover, by the equality above, we see that $\lim_{t \rightarrow 0} Ad_{1+t^{-1}v_0}^*(t\phi)|_{\mathfrak{n}} = 0$; thus it can be regarded as an element of $(\mathfrak{g}/\mathfrak{n})^* = \mathfrak{l}^*$.

Furthermore, we have

$$\lim_{t \rightarrow 0} Ad_{1+t^{-1}v_0}^*(t\phi)|_{[\mathfrak{l}, \mathfrak{l}]} = -\bar{\phi}_{v_0},$$

which can be regarded as an element of $[\mathfrak{l}, \mathfrak{l}]$ by the identification $[\mathfrak{l}, \mathfrak{l}]^* \cong [\mathfrak{l}, \mathfrak{l}]$. As remarked above, this is not a nilpotent element.

Let us write \mathfrak{l} as a direct sum of the semi-simple part and the center: $\mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] \oplus z(\mathfrak{l})$. There is an L -equivariant isomorphism $\mathfrak{l}^* \cong [\mathfrak{l}, \mathfrak{l}]^* \oplus z(\mathfrak{l})^*$. Here L acts trivially on the second factor $z(\mathfrak{l})^*$. Therefore, every coadjoint orbit of \mathfrak{l}^* is a pair of a coadjoint orbit of $[\mathfrak{l}, \mathfrak{l}]^*$ and an element of $z(\mathfrak{l})^*$.

In our situation, we can write

$$\bar{\phi}([v_0, \cdot]) = \bar{\phi}_{v_0} \oplus \bar{\phi}([v_0, \cdot])|_{z(\mathfrak{l})}.$$

We can apply the same argument for $\lambda\phi$ with an arbitrary $\lambda \in \mathbf{C}^*$ to conclude that $\lambda\bar{\phi}([v_0, \cdot]) \in \bar{O}$. One can write

$$\lambda \cdot \bar{\phi}([v_0, \cdot]) = \lambda\bar{\phi}_{v_0} \oplus \lambda \cdot \bar{\phi}([v_0, \cdot])|_{z(\mathfrak{l})}.$$

Since $\bar{\phi}_{v_0}$ is not an nilpotent element, we see that $\lambda\bar{\phi}_{v_0}$ ($\lambda \in \mathbf{C}^*$) are contained in mutually different coadjoint orbits of $[\mathfrak{l}, \mathfrak{l}]^*$.

Therefore, $\lambda \cdot \bar{\phi}([v_0, \cdot])$ ($\lambda \in \mathbf{C}^*$) are also contained in mutually different coadjoint orbits of \mathfrak{l}^* . Q.E.D.

Proof of Theorem. We already know that $wt(\omega) = 1$ or $wt(\omega) = 2$. In the latter case (X, ω) is isomorphic to $(\mathbf{C}^{2d}, \omega_{st})$. So we assume that $wt(\omega) = 1$. By Propositions 2, 3 and 4, (X, ω) is isomorphic to a coadjoint orbit closure (\bar{O}, ω_{KK}) of a complex reductive Lie algebra \mathfrak{g} together with the Kirillov-Kostant form. Assume that \mathfrak{g} is not semisimple, i.e., \mathfrak{g} has non-trivial center $z(\mathfrak{g})$. In our case O is preserved by the scalar \mathbf{C}^* -action on \mathfrak{g}^* . Such an orbit is contained in $(\mathfrak{g}/z(\mathfrak{g}))^*$. This contradicts the fact that $T_0\bar{O} = \mathfrak{g}^*$. Hence \mathfrak{g} is semisimple. For a semisimple Lie algebra, a coadjoint orbit is identified with an adjoint orbit by the Killing form. A coadjoint orbit preserved by the scalar \mathbf{C}^* -action corresponds to a nilpotent orbit by this identification.

References

- [Be] Beauville, A. : Symplectic singularities, Invent. Math. **139** (2000), 541-549
- [Hi] Hinich, V. : On the singularities of nilpotent orbits, Israel J. Math. **73** (1991), 297-308
- [Ho] Hochschild, G. P.: Basic theory of algebraic groups and Lie algebras, Graduate Texts in Mathematics **75** (1981)
- [Ka] Kaledin, D.: Symplectic varieties from the Poisson point of view, J. Reine Angew. Math. **600** (2006), 135-160
- [K-P] Kraft, H., Procesi, C.: On the geometry of conjugacy classes in classical groups, Comment Math. Helv. **57** (1982), 539-602
- [Mos] Mostow, G.D.: Fully reducible subgroups of algebraic groups, Amer. J. Math. **78** (1956) 200-221
- [Na 1] Namikawa, Y.: A finiteness theorem on symplectic singularities, Compositio Math. **152** (2016), 1225-1236
- [Na 2] Namikawa, Y.: On the structure of homogeneous symplectic varieties of complete intersection, Invent. Math. **193** (2013) 159-185
- [Na 3] Namikawa, Y.: Equivalence of symplectic singularities, Kyoto Journal of Mathematics, **53**, No.2 (2013), 483-514
- [Na 4] Namikawa, Y.: Flops and Poisson deformations of symplectic varieties, Publ. Res. Inst. Math. Sci. **44** (2008), 259 - 314
- [Pa] Panyushev, D. I. : Rationality of singularities and the Gorenstein property of nilpotent orbits, Funct. Anal. Appl. **25** (1991), 225-226
- [Pro] Procesi, C.: Lie groups, an approach through invariants and representations, (2007), UTX, Springer

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